Dimension of interaction dynamics

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(February 8, 2008)

A method allowing to distinguish interacting from non-interacting systems based on available time series is proposed and investigated. Some facts concerning generalized Renyi dimensions that form the basis of our method are proved. We show that one can find the dimension of the part of the attractor of the system connected with interaction between its parts. We use our method to distinguish interacting from non-interacting systems on the examples of logistic and Hénon maps. A classification of all possible interaction schemes is given.

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I. INTRODUCTION

Given two time series can one tell if they originated from interacting or non-interacting systems? We show that with the help of embedding methods [1], Takens theorem [2,3] and some facts concerning Renyi dimensions which we prove, one can succeed in case of chaotic systems. Moreover, one can quantify the common part of the dynamics, which we call dynamics of interaction.

It happens sometimes, especially in simple systems like electronic circuits or coupled mechanical oscillators, that one knows whether the systems under investigation are coupled or not, what is the direction and sometimes strength of the coupling. However, there are many complex phenomena in nature where one is unable to verify directly the existence of coupling between parts of the system in which the phenomenon takes place.

Especially in complex spatiotemporal systems, like fluid systems, brain, neuronal tissue, social systems etc. one often faces the problem of characterization of interdependence of parts of the system of interest and quantifying the strength of interactions between the parts.

Recent research in neurology, for example, has shown that temporal coordination between different, often distant neural assemblies plays a critical role in the neurophysiological underpinnings of such cognitive phenomena as the integration of features in object representation (cf. [4] for a review) and the conscious experience of stimuli [5]. The critical empirical question, therefore, is which of the neural assemblies synchronize their activity. Since coordination may take many forms, including complex non-linear relations, simple correlational methods may not be sufficient to detect it. The detection of nonlinear forms of coordination is also critically important for issues in cognitive science [6], developmental psychology [7], and social psychology [8,9].

The method traditionally used for this purpose is correlation analysis. Given two time series one studies their autocorrelation functions and cross-correlations. Large cross-correlations are usually attributed to large interdependence between the parts. Small cross-correlations are considered as the signature of independence of the variables.

Unfortunately, the linear time series analysis gives meaningful results only in case of linear systems or stochastic time series. It is well-known that spectral analysis alone cannot discriminate between low-dimensional nonlinear deterministic systems and stochastic systems [10], even though the properties of the two kinds of systems are different.

Methods based on entropy measures represent one viable approach for detecting nonlinear relations between the activity of different neural assemblies [5].

Recently another approach based on nonlinear mutual prediction has been proposed and used in an experiment. Pecora, Caroll and Heagy [11] developed a statistics to study the topological nature of functional relationship between coupled systems. Schiff et al. [12] used it as a basis of their method. The idea is as follows: if there exists a functional relationship between two systems, it is possible to predict state of one system from the known states of the other. This happens if the coupling between two systems is strong enough so that generalized synchronization occurs [11,13,14]. The average normalized mutual prediction error is used to quantify the strength and directionality of the coupling [12].

The method we introduce in the present paper does not assume generalized synchrony. We introduce the notion of the *dimension of interaction*, which measures the size of the dynamics responsible for the coupling between the two systems. More precisely, it is the dimension of the part of the attractor of the whole system, which is acted on by the dynamics of both subsystems. We also show how to obtain information concerning the strength and directionality of the coupling.

The idea is, in fact, very simple. Given two time series from subsystems of interest we construct another one which probes the whole system, for instance adding the two series. If the subsystems do not interact, dimension

of the whole system is the sum of the dimensions of the two subsystems, all of which can be estimated from data. On the other hand, if the subsystems have some common degrees of freedom, dimension of the whole system will be smaller than the sum of the dimensions of the two subsystems.

Our method can also be used to find out if two response systems have a common driver. We discuss this application in Section III.

The structure of the paper is as follows. In Section II we recall the definition of the Renyi dimensions and formulate three theorems which form the basis of our method. The, rather straightforward, proofs have been relegated to Appendix A, since they are not crucial for understanding the method itself and can be omitted by readers whose main interest is in applications. We formulate our method in section III. Classification of all the possible interaction schemes is given in Section IV. A simple way of verifying the kind and direction of the coupling is provided. Results from the simulations of coupled logistic and Hénon maps are collected in Section V. Final comments and outlook are given in the last section.

II. THEORETICAL CONSIDERATIONS

Our method presented in Section III is based on three theorems relating dimensions of subsystems to the dimension of the whole system. The first one states the intuitively obvious fact that the dimension of a system consisting of two non-interacting parts is the sum of the dimensions of the subsystems. A less trivial Theorems 2 and 3 establish interdependencies among the dimensions of the system and its interacting parts. Before we state our theorems we shall recall the definition of the Renyi dimensions.

A. Renyi dimensions

It is at present generally accepted that a lot of objects, both in the real physical space and in the phase space, are multifractals [15–18]. This means they can be described by (statistically) self-similar probability measures. This usually implies that they can be decomposed into a (infinite) number of objects of different Haussdorff dimensions, or, equivalently, they have non-trivial multifractal spectra of dimensions.

The Renyi dimensions [19] have drawn attention of physicists and mathematicians after publication of the papers by Grassberger, Hentschel and Procaccia [20–23]. For a probability measure μ on a d-dimensional space U one takes a partition of U into small cells of equal linear size ε (equal volume ε^d). One defines the Renyi dimensions[1] as

$$D_q(\mu) := \begin{cases} \lim_{\varepsilon\to 0} \frac{1}{q-1} \frac{\log \sum_i p_i^q}{\log \varepsilon}, & \text{for } q \in \mathbb{R} \setminus \{1\} \\ \lim_{\varepsilon\to 0} \frac{1}{q-1} \frac{\sum_i p_i \log p_i}{\log \varepsilon}, & \text{for } q = 1, \end{cases} \tag{1}$$

where

$$p_i = \mu(i - \text{th cell}) = \int_{i-\text{th cell}} d\mu(x),$$

and the sum is taken over all cells with $p_i \neq 0$.

Of particular importance are D_0 — the box-counting dimension, usually equal to the Hausdorff dimension [15,24,25], D_1 — the information dimension or the dimension of the measure [19,26–29], which describes how the entropy $-\sum_i p_i \log p_i$ increases with the change of the scale, and D_2 — the correlation dimension [22,23,30], which can be most easily extracted from data, usually treated as a lower estimate of D_1 since $D_{q_1} \leq D_{q_2}$ for $q_1 > q_2$.

Generalized dimensions are defined for all real q, however in proofs we shall restrict our attention to the case $q \geq 1$. We are particularly interested in $q = 1$ and $q = 2$.

B. Non-interacting systems

Consider two non-interacting dynamical systems (U_1, φ_1, μ_1), (U_2, φ_2, μ_2), where $U_i \subset \mathbb{R}^{n_i}$ is the phase space, φ_i is a flow or a map on U_i, and μ_i is an ergodic φ_i-invariant natural measure on U_i.

Below we shall concentrate on the case of continuous systems. Changes needed for the discrete time case are mostly notational.

By natural measure μ we mean

$$\mu = \mu(x_0) := \lim_{T\to\infty} \frac{1}{T} \int_0^T \delta(x - \varphi_t(x_0))dt; \tag{2}$$

in the weak sense for μ-almost every x_0 (one typically thinks of some physical measure, like Sinai-Ruelle-Bowen measure [40]); $\mu_i(U_i) = 1$. The limit in the weak sense means that if we integrate $\mu(x_0)$ with a continuous function f on U the limit (2) exists and is μ-almost everywhere independent of x_0, or in other words, the average of f along a typical trajectory is independent of the trajectory, thus time averages are equal to ensemble averages.

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[1] An equivalent description of multifractal measures is f_α spectrum [28,31,33]. A thorough discussion of the properties of D_q and f_α spectra falls beyond the scope of this paper. Some good reviews of these with the discussion of the abundant literature on multifractals can be found e.g. in [16–18,34–39]. Mathematically precise definitions of multifractal spectra can be found in [17,18].

We are interested in such measures that the set of x_0 for which $\mu(x_0) = \mu$ has a non-zero Lebesgue measure.

The composite non-interacting system has a product structure $(U_1 \times U_2, \varphi_1 \times \varphi_2, \mu_1 \times \mu_2)$. Its dynamics can be written as

$$\begin{cases} \mathbf{u}_1(t) &= \varphi_1(\mathbf{u}_1(0), t), \\ \mathbf{u}_2(t) &= \varphi_2(\mathbf{u}_2(0), t). \end{cases}$$

Theorem 1 *Suppose $D_q(\mu_1), D_q(\mu_1), D_q(\mu_1 \times \mu_2)$ exist. Then*

$$D_q(\mu_1 \times \mu_2) = D_q(\mu_1) + D_q(\mu_2). \tag{3}$$

This means, as should be intuitively obvious, the dimensions of non-interacting subsystems add up to the dimension of the whole system. The proof is given in Appendix A. It follows from extensivity of Renyi entropies.

This is in fact one of the long-standing problems in the dimension theory, namely finding the conditions under which the equality holds for various dimensions for arbitrary measures. Some results for Olsen's version of multifractal formalism with a discussion of previous results can be found in [41].

C. Interacting systems

Take two interacting subsystems U_1 and U_2 of system U. It may happen that all the variables in U_1 couple with all those in U_2 but this is not necessary. For many-dimensional systems the structure of the equations of dynamics can be very complicated.

Consider the following decomposition of variables of U_i. Let $\mathbf{y}_1$ be the largest set of variables in U_1 satisfying the condition that if you change their state whatsoever, it will not influence the future evolution of U_2. Similarly define $\mathbf{y}_2$. Put all the other variables of U_1, U_2 in vector $\mathbf{x}$. They form a dynamical system V — the part of the whole system which is responsible for the interaction. Then the dynamics of the whole system U can be written as

$$\begin{cases} \dot{\mathbf{x}} &= f(\mathbf{x}), \\ \dot{\mathbf{y}}_1 &= g_1(\mathbf{x}, \mathbf{y}_1), \\ \dot{\mathbf{y}}_2 &= g_2(\mathbf{x}, \mathbf{y}_2). \end{cases} \tag{4}$$

Thus the dynamics of the interacting systems U_1 and U_2 is formally equivalent to dynamics of three systems: X (interaction part) driving Y_1 and Y_2. We pursue this analogy deeper in the next section. An example when such decomposition arises naturally is given in Appendix B.

Let $\mu_U, \mu_1, \mu_2, \mu_V$ be natural measures of dynamical systems, respectively, U, U_1, U_2, V.

Theorem 2 *Suppose $D_1(\mu_1)$, $D_1(\mu_2)$, $D_1(\mu_V)$, $D_1(\mu_U)$ exist. Then*

$$D_1(\mu_V) \le d_{\mathsf{int}} := D_1(\mu_1) + D_1(\mu_2) - D_1(\mu_U). \tag{5}$$

(We shall call d_{int} dimension of interaction*). The equality holds when $\mathbf{y}_1$ and $\mathbf{y}_2$ are asymptotically independent.*

Asymptotical independence means essentially lack of generalized synchronization between the $\mathbf{y}$s and their common driver $\mathbf{x}$. We relegate further discussion to Appendix A, where we make this condition precise and show where it is needed[2].

If we think of dimensions as estimates on the number of degrees of freedom, the Theorem 2 means intuitively that if the system can be considered as composed of interacting parts, some of the degrees of freedom — perhaps even all — are common for both of the parts. Therefore, dimension of the whole system is equal to the sum of the number of the common degrees of freedom, those degrees of freedom which belong to U_1 and do not belong to U_2, and the other way round. Thus if we add the dimensions of the subsystems U_1 and U_2, we count the common degrees of freedom twice. We must therefore subtract them if we want to get dimension of the whole system U.

In the above theorem we show that this intuition can be made precise *only* in case of the information dimension D_1 and with an additional assumption. The notion of the dimension of interaction we define in equation (5) is crucial for our method.

In the special case, when one (or both) of $U_i = V$, (all the variables of U_i couple with some of the variables of the other subsystem), say $U_2 = V$, we may establish

Theorem 3 *Suppose $D_q(\mu_1)$, $D_q(\mu_2)$, $D_q(\mu_V)$ exist and $k_2 = n_2$. Then*

$$D_q(\mu_V) = d_q^{\mathsf{int}} := D_q(\mu_1) + D_q(\mu_2) - D_q(\mu_U). \tag{6}$$

The proof is obvious, for in this case $U_2 \equiv V$ and $U_1 = U$. This also means that the above intuitions in this case are precise for arbitrary generalized dimensions and no further assumptions are needed.

The generalized dimensions of interaction d_q^{int} are estimates on the number of effective degrees of freedom responsible for the interaction between the parts of the system under study. Of most interest are $d_1^{\mathsf{int}} \equiv d_{\mathsf{int}}$, which has the best analytical properties, and d_2^{int}, which can be most reliably estimated from data.

Note that

$$\begin{aligned} \max\{d_q^{\mathsf{int}}, D_q(\mu_V)\} &\le \min\{D_q(\mu_1), D_q(\mu_2)\} \\ &\le \max\{D_q(\mu_1), D_q(\mu_2)\} \\ &\le D_q(\mu_U). \end{aligned} \tag{7}$$

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[2] One would like to establish a similar inequality in case of other Renyi dimensions, however, in general

$$D_q(\mu_V) + D_q(\mu_U) - D_q(\mu_1) - D_q(\mu_2)$$

can have arbitrary sign (cf. Appendix A). Nevertheless, we expect this difference for typical physical systems to be small in comparison with the dimensions involved.

Furthermore, for $q = 1$ one can show

$$0 \leq D_1(\mu_V) \leq d_1^{\mathsf{int}}.$$

We conjecture $d_q^{\mathsf{int}} \geq 0$ also for $q > 1$.

III. THE METHOD

Suppose we are given two time series measured in subsystems U_1 and U_2 of system U whose structure and interdependence we do not know, e.g. signals gathered on two electrodes placed in not too far away portions of brain, or measurements of velocity or temperature in various parts of moderately turbulent fluid. We would like to know, if the equations governing the dynamics of both of these variables are coupled or not, how many degrees of freedom are common and what is the direction of the coupling.

Let X_i be a function on U_i, i.e. $X_i : U_i \to \mathbb{R}$. The time series we measure are $x_1(n) := X_1(\mathbf{u}_1(t_n))$ and $x_2(n) := X_2(\mathbf{u}_2(t_n))$. Let $Y : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function nontrivially depending on both variables[3]. We construct another time series $y(n) = Y(x_1(n), x_2(n))$. Thus $Y(X_1, X_2)$ is a function on U.

Using time delay method [1,2] we can reconstruct the dynamics of the systems U_i and U from $x_i(n)$ and $y(n)$. Namely, for a given *delay* τ and *embedding dimension* N we construct *delay vectors*

$$\tilde{\mathbf{u}}_1(n) = (x_1(n), x_1(n-\tau), \ldots, x_1(n-(N-1)\tau));$$

the construction of $\tilde{\mathbf{u}}_2$ from x_2 and $\tilde{\mathbf{u}}$ from y is similar.

If $N > 2D_0(\mu_1)$, for all reasonable delays, for infinite not-too-sparsely probed time series, the Takens theorem [2,3] guarantees $\tilde{\mathbf{u}}_1(n)$ is an embedding of the original invariant set in U_1. To calculate dimensions it is even enough to take $N > D_0(\mu_1)$ [42,43]. It is generally believed that also for finite but not too short and not too noisy time series the above construction gives occasionally a reasonable estimate on the original dynamics. For a detailed discussion of these issues the reader should consult the relevant literature, e.g. [10,44–47]. We disregard the practical problems until section V where we show some numerical results. For the time being we discuss clean infinite time series.

Having reconstructed the attractors we can estimate their generalized dimensions and calculate the *generalized dimensions of interaction*

$$d_q^{\mathsf{int}} := D_q(\mu_1) + D_q(\mu_2) - D_q(\mu_U). \tag{8}$$

It is also convenient to consider normalized dimensions of interaction:

$$\begin{aligned} m_1^q &:= d_q^{\mathsf{int}}/D_q(\mu_1), \\ m_2^q &:= d_q^{\mathsf{int}}/D_q(\mu_2), \\ m_U^q &:= d_q^{\mathsf{int}}/D_q(\mu_U). \end{aligned} \tag{9}$$

From the values of m_i^q we can infer the information we need. All the possible cases are described in the next section. Note that if $m_i^q \neq 0$, they satisfy

$$\frac{1}{m_1^q} + \frac{1}{m_2^q} - \frac{1}{m_U^q} = 1.$$

From (7) we also have

$$0 \leq m_U^q \leq m_1^q, m_2^q \leq 1,$$

which provides us with a tool to check consistency of results.

Before we present the classification of all the possible schemes of interaction let us discuss heuristically four simple examples.

I If U_1, U_2 are uncoupled, the variables we see through x_1, x_2 are different, thus $\mu_U = \mu_1 \times \mu_2$. Therefore, from Theorem 1, $d_q^{\mathsf{int}} = 0$, as it should be for any reasonable definition of dimension of interaction for non-interacting systems.

II Consider now a system U consisting of three isolated systems V_i, which we cannot observe separately, however, but rather through U_1 and U_2, e.g. measuring $X_1(\mathbf{v}_1, \mathbf{v}_2)$ and $X_2(\mathbf{v}_2, \mathbf{v}_3)$. Reconstructing dynamics from time series of X_1

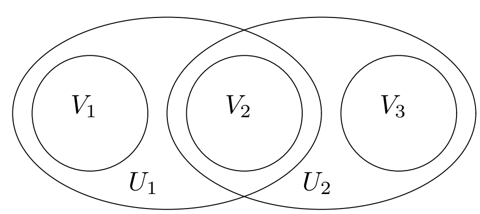


FIG. 1. Simple interaction

and X_2 we expect to obtain

$$\begin{aligned} D_q(\mu_1) = D_q(\mu_{V_1}) + D_q(\mu_{V_2}), \\ D_q(\mu_2) = D_q(\mu_{V_2}) + D_q(\mu_{V_3}). \end{aligned}$$

With a typical function $Y(x_1, x_2)$ we obtain time series $y(n)$ from which we estimate

[3] For finite noisy time series some functions are better than other. In practice we used five different functions $Y(x, y)$, namely $x+y$, $x \cdot y$, $\sin(x)\cos(y)$, $x\exp(y)$, $2x-y$, to calculate dimension of the system $D_q(\mu_U)$, and averaged the results. The variance of the obtained five estimates was usually small. The above functions were not chosen for their particularly good numerical properties but rather to verify that the results obtained depend only weekly on the choice of the function Y.

$$D_q(\mu_U) = D_q(\mu_{V_1}) + D_q(\mu_{V_2}) + D_q(\mu_{V_3}).$$

Since dynamics of V_2 is responsible for the interaction between U_1 and U_2, we want to call the *dimension of interaction* dimension of μ_{V_2}. According to the definition (8) we have

$$\begin{aligned} d_q^{\mathsf{int}} &= D_q(\mu_1) + D_q(\mu_2) - D_q(\mu_U) \\ &= D_q(\mu_{V_1}) + D_q(\mu_{V_2}) + D_q(\mu_{V_2}) + D_q(\mu_{V_3}) + \\ &\quad -[D_q(\mu_{V_1}) + D_q(\mu_{V_2}) + D_q(\mu_{V_3})] \\ &= D_q(\mu_{V_2}). \end{aligned}$$

III Consider now the general situation described in section II C. Reconstructing dynamics from time series of typical variables from systems U_1 and U_2, say $x_1(n)$ and $x_2(n)$ we get

$$\begin{aligned} D_q(\mu_1) &\geq D_q(\mu_V), \\ D_q(\mu_2) &\geq D_q(\mu_V). \end{aligned}$$

For a typical function $Y(x_1, x_2)$ we obtain

$$\begin{aligned} &\max\{D_q(\mu_1), D_q(\mu_2)\} \\ &\leq D_q(\mu_U) \\ &\leq D_q(\mu_V) + (D_q(\mu_1) - D_q(\mu_V)) + \\ &\quad (D_q(\mu_2) - D_q(\mu_V)), \end{aligned}$$

where $D_q(\mu_1) - D_q(\mu_V)$ quantifies number od degrees of freedom in U_1 not coupled to U_2. From this we conclude

$$\begin{aligned} 0 < D_q(\mu_V) &\leq d_q^{\mathsf{int}} \\ &= D_q(\mu_1) + D_q(\mu_2) - D_q(\mu_U) \\ &\leq \min\{D_q(\mu_1), D_q(\mu_2)\}, \end{aligned}$$

the difference between $D_q(\mu_V)$ and d_q^{int} depending on the strength of synchronization between U_1 and U_2.

IV As the last example we shall take a system X driving two response systems Y_1 and Y_2. Suppose we also have a second copy of this setup, namely drive X' with response systems Y_1' and Y_2'. We collect simultaneously four time series of some variable from all of the response systems. Now we choose randomly two of them and want to know if the systems they come from had a common driver.

It is easy to check that if they had, then d_q^{int} is approximately the dimension of the driver $D_q(\mu_X) > 0$. If they had different drivers, then $d_q^{\mathsf{int}} = 0$.

Summarizing, from measurements involving parts of the given system and arbitrary nontrivial smooth function of two variables we can reconstruct the dimensions of measures μ_1, μ_2 and μ_U. From this we can obtain the dimension of interaction d_q^{int} (8). Depending on the values of $D_q(\mu_1), D_q(\mu_2), D_q(\mu_U)$ and d_q^{int} we can find out if the systems are coupled or not, and what is the direction of coupling.

IV. CLASSIFICATION OF POSSIBLE INTERACTION SCHEMES

Let us thus assume that we have two subsystems and the reconstructed dimensions for them are $D_q(\mu_1)$ and $D_q(\mu_2)$. The dimension of the whole system $D_q(\mu_U)$ is obtained from time series $y(n)$ constructed through the procedure described in the previous section. The dimension of interaction is calculated from (8). The above discussion leads to a question which situations are possible. There are four non-equivalent cases, which are conveniently described by the following proposition

Proposition 4.

1. *If $d_q^{\mathsf{int}} = 0$, then $\mu_U = \mu_1 \times \mu_2$ (the systems U_1 and U_2 do not interact);*
2. *If $D_q(\mu_1) = D_q(\mu_2) = d_q^{\mathsf{int}}$, then $\mu_U = \mu_1 \equiv \mu_2 \equiv \mu_V$ (the systems U_1 and U_2 are the same system or we have maximal coupling) ;*
3. *If $D_q(\mu_1) > D_q(\mu_2) = d_q^{\mathsf{int}}$, then $\mu_2 = \mu_V$ and $\mu_1 \equiv \mu_U$ (all variables of U_2 couple to some of the degrees of freedom of U_1, or U_2 is the driver in the pair driver—response which is $U_1 \equiv U$);*
4. *In all the other cases $D_q(\mu_1), D_q(\mu_2) > d_q^{\mathsf{int}}$, which means interaction or double control (two response systems driven by a common driver).*

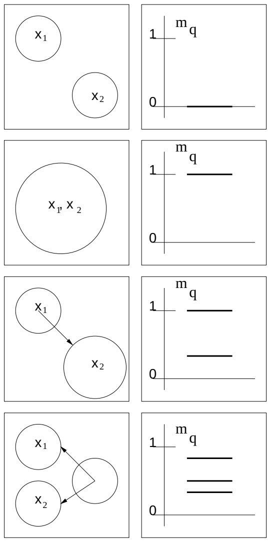
Note that this proposition is to some extent opposite to the theorems proposed in Section II. It can be shown for $q = 1$ [48]. We verify it numerically for particular systems for $q = 2$ in the next section.

It is convenient sometimes to use m_1^q, m_2^q, m_U^q (9). We can write the above classification in this case as follows

1. $m_1^q = m_2^q = m_U^q = 0$ means no interaction;
2. $1 = m_1^q = m_2^q = m_U^q$ means maximal coupling: $\mu_1 \equiv \mu_2 \equiv \mu_V$;
3. $1 = m_1^q > m_2^q = m_U^q > 0$ means coupling of all the degrees of freedom of $U_2 \equiv V$ to some variables of U_1;
4. $1 > m_1^q \geq m_2^q > m_U^q > 0$ means interaction or double control (two systems driven by a common driver);

All the four cases are presented symbolically on Figure 2.

The examples considered in the previous section can be easily identified as particular cases of this classification. Namely example I represents case 1, example II represents case 4, example III can represent cases 2, 3 or 4, the last example represents cases 1 or 4 depending on whether the signals analyzed come from systems coupled to the same driver or not.



$$d_q^{\mathrm{int}} = 0$$

$$m_1^q = m_2^q = m_U^q = 0$$

$$D_q(\mu_1) = D_q(\mu_2) = d_q^{\mathrm{int}} > 0$$

$$m_1^q = m_2^q = m_U^q = 1$$

$$D_q(\mu_2) > D_q(\mu_1) = d_q^{\mathrm{int}} > 0$$

$$1 = m_2^q > m_1^q = m_U^q > 0$$

$$D_q(\mu_1), D_q(\mu_2) > d_q^{\mathrm{int}} > 0$$

$$1 > m_1^q \geq m_2^q > m_U^q > 0$$

FIG. 2. Classification of possible interaction schemes. The first column shows symbolically the relative position in the phase space of the subsystems in which we measure the time series x_1 and x_2. An arrow from one system to another means the future states of the second system depend on the current states of both. The second column shows the values of normalized dimensions m_1^q, m_2^q and m_U^q in each of the cases.

V. NUMERICAL RESULTS

Below we shall present some applications of our method to analysis of numerical results for several paradigmatic systems (coupled Hénon maps and logistic maps).

Throughout this section we will use d_2^{int}. The dimensions presented in the pictures are always D_2 calculated with the help of `d2` program from TISEAN package [49] with an algorithm which is an extension of algorithms published previously [22,23,30] which improves speed of computation [49]. In every case we used 10^5 points with one exception described in the text. The functions Y used to calculate the dimension of the whole system (cf. previous sections) were $x+y$, $x \cdot y$, $\sin(x)\cos(y)$, $x\exp(y)$, $2x-y$. To estimate the dimension we used Takens-Theiler estimator [10,49–51] `c2t` and `c2d` smoothed output from `d2`.

Typical behavior of local dimension $d \log C(\varepsilon)/d \log \varepsilon$ as a function of resolution ε is shown in figure 3.

A. Two Hénon maps

Consider a system U consisting of two Hénon maps [52] coupled as follows [12]:

$$K \begin{cases} x_{i+1} &= 1.4 - x_i^2 + 0.3 y_i, \\ y_{i+1} &= x_i, \end{cases}$$

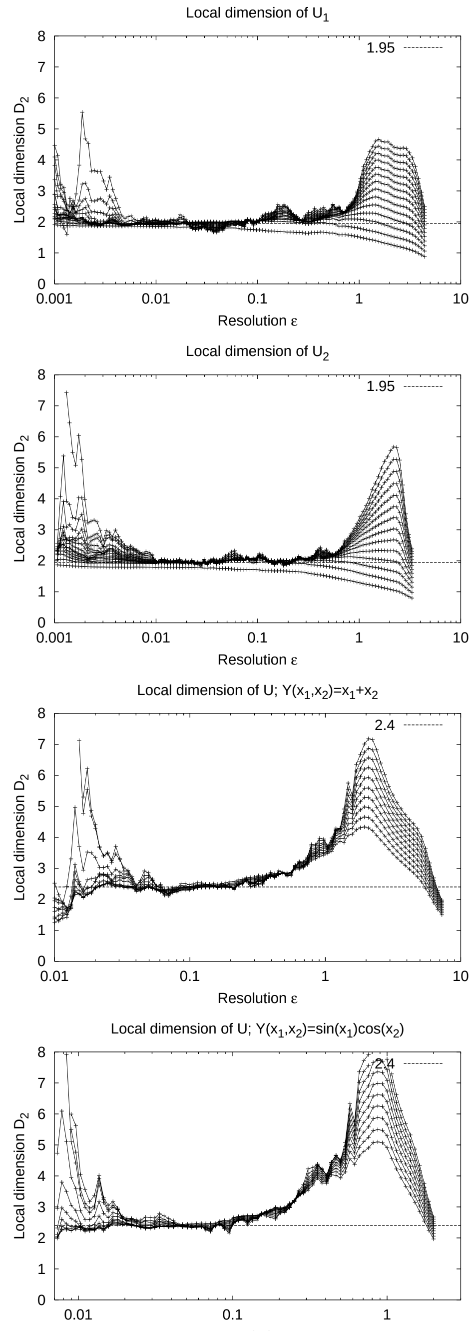


FIG. 3. Takens estimator of correlation dimension. Data shown come from two different Hénon systems driven by the third (eq. 11 with parameters $B_1 = 0.3$, $C_1 = 0.5$, $B_2 = 0.1$, $C_2 = 0.6$.). Correlation dimension estimated from the pictures is 1.95 for both of the subsystems, and 2.4 for the whole system. We show two plots out of five used to estimate the last number. Dimension of interaction in this case is $1.95 + 1.95 - 2.4 = 1.5 > 1.22$, which suggests partial synchronization of the two response systems with the driver.

$$L\left\{\begin{array}{rcl} u_{i+1} &=& 1.4-(Cx_i+(1-C)u_i)u_i+Bv_i,\\ v_{i+1} &=& u_i, \end{array}\right. \qquad (10)$$

Thus Hénon system K drives system L. The coupling is introduced through variable u. We consider the case of coupled identical systems ($B = 0.3$) and non-identical coupled systems ($B = 0.1$). Parameter C measures the strength of interaction.

Suppose the variables accessible experimentally are x_n and u_n. What can be said in this case about the interaction between systems K and L?

Certainly, for $C = 0$ the systems K and L do not interact (case 1. in our classification), therefore $D_q(\mu_U) = D_q(\mu_K) + D_q(\mu_L)$ and $d_q^{\mathsf{int}} = 0$. On the other hand, for positive C the influence of x should reflect in the behavior of u. From Theorem 3 we expect $D_q^{\mathsf{int}} = D_q(\mu_K)$ (case 3.). One can also expect for C raising slightly above 0, $D_q(\mu_U)$ not to change much, while $D_q(\mu_L)$ should jump from its value at 0 to the value of $D_q(\mu_U)$ at $c = 0$.

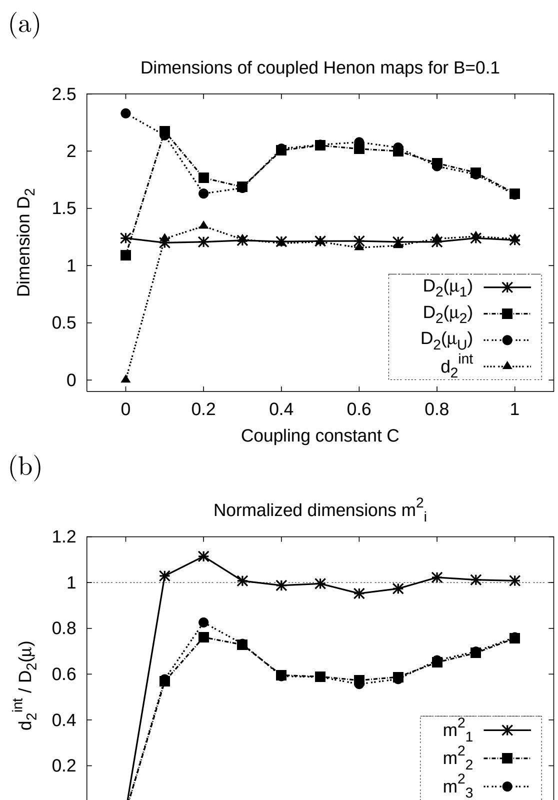


FIG. 4. a) Dimensions $D_2(\mu_1), D_2(\mu_2), D_2(\mu_U)$ and d_2^{int} of one-way coupled non-identical Hénon maps (10) $B = 0.1$. b) Normalized dimensions m_1^2, m_2^2 and m_U^2 for the same systems.

This behavior can indeed be seen in figure 4a for non-identical Hénon systems ($B = 0.1$) and in 5a for identical systems ($B = 0.3$). The synchronization of x and u [53]

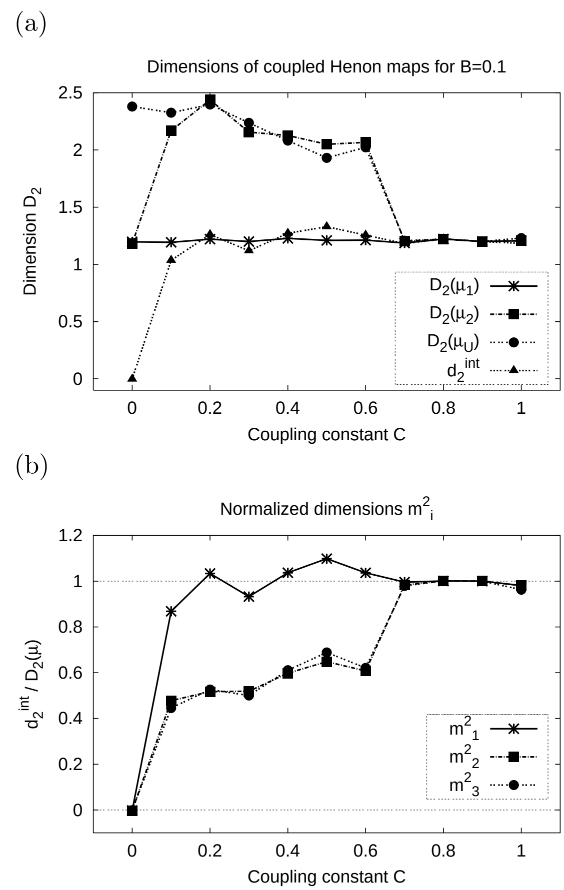


FIG. 5. a) Dimensions $D_2(\mu_1), D_2(\mu_2), D_2(\mu_U)$ and d_2^{int} of one-way coupled identical Hénon maps (10) $B = 0.3$. b) Normalized dimensions m_1^2, m_2^2 and m_U^2 for the same systems.

visible for $C \geq 0.7$ (case 2.) can be discovered much simpler, namely if one plots several consecutive values of $x_n - u_n$ (100, say) versus coupling, for these particular values all of the points fall on 0 (cf. figure 7 of [12]).

Looking at the normalized dimensions (fig. 4b and 5b) we easily identify lack of coupling for $C = 0$ ($m_1 = m_2 = m_3 = 0$), case 4. (maximal coupling) for $B = 0.3$ and $C \geq 7$ ($m_1 = m_2 = m_3 = 1$), and case 3. in all the other cases.

The drop-down of the dimension at 0.7 for identical systems is connected with the full synchronization of the systems. The equations (10) admit solutions symmetric in x and u ($x_n - u_n = 0$), which at this region become stable and the whole probability measure gets localized on a lower-dimensional manifold. For more details cf. [12].

B. Three Hénon maps

Consider now the system U consisting of three Hénon maps [52] coupled as follows [12]:

$$
\begin{aligned}
K&\begin{cases} x_{i+1} = 1.4 - x_i^2 + 0.3y_i, \\ y_{i+1} = x_i, \end{cases}\\
L&\begin{cases} u_{i+1} = 1.4 - (C_1 x_i + (1 - C_1)u_i)u_i + B_1 v_i, \\ v_{i+1} = u_i, \end{cases}\\
M&\begin{cases} w_{i+1} = 1.4 - (C_2 x_i + (1 - C_2)w_i)w_i + B_2 z_i, \\ z_{i+1} = w_i. \end{cases}
\end{aligned} \tag{11}
$$

Thus Hénon system K drives systems L and M. The coupling is introduced through variables u and w. Parameters C_1, C_2 measure the strength of interaction.

Suppose the measurements on (K, L, M) yield variables u and w. What can be said in this case about the interaction between the systems L and M?

For $C_1 = C_2 = 0$ neither L nor M systems feel the influence of K. They also do not interact (case 1.). When one of C_i grows, the influence of K is immediately mirrored in the rise of the dimension of μ_L or μ_M. For both $C_i > 0$ the systems L and M interact (case 2.), and the part responsible for interaction is K. Thus the dimension of the common part is constant and equal to 1.22 in our case.

We show this behavior in figures 6a for different parameters ($B_1 = 0.3, B_2 = 0.1$) and 7a for the same parameters ($B_1 = 0.3, B_2 = 0.3$). In both cases $C_1 = 0.5$ and C_2 is varied. In both figures one can clearly see the jump of the dimension of interaction from 0 to values equal to or greater than 1.22, the dimension of the attractor of K.

Figure 7 is particularly interesting, since one can apparently identify all the four cases from our classification. For $C_2 = 0$ we have non-interacting systems, for $C_2 \in [0.2, 0.4]$ and $C_2 = 0.6$ we have case 2.

For $C_2 = 0.5$ the two Hénon systems L and M become identical. Since at this value of coupling constant they are in general synchrony with the driver, which means

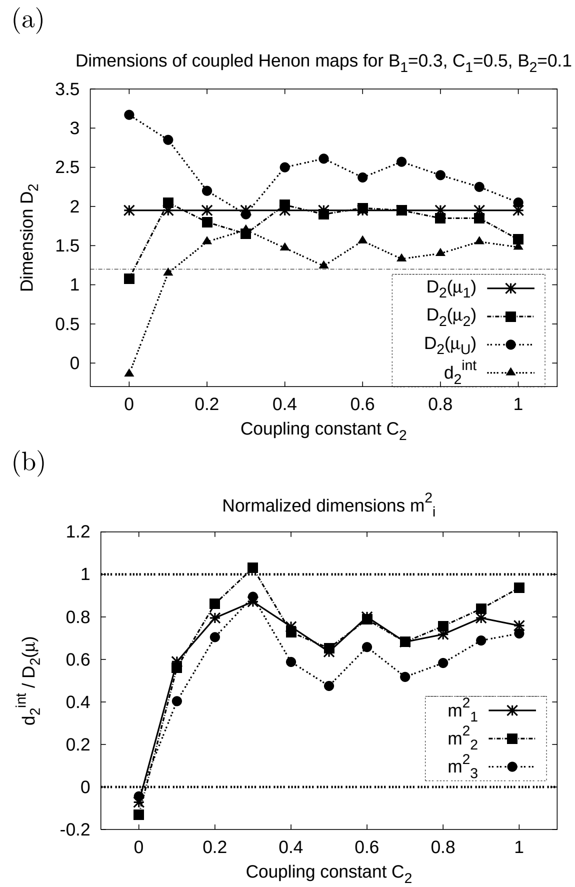
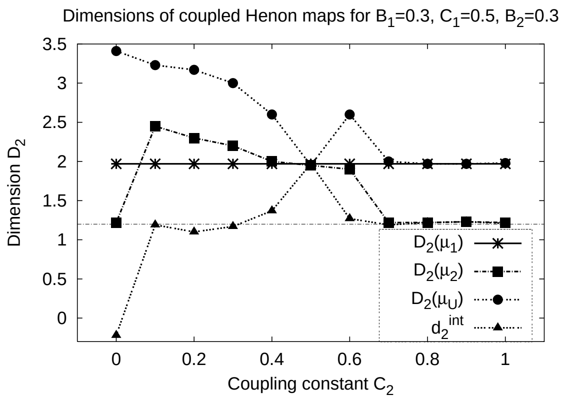


FIG. 6. a) Dimensions $D_2(\mu_1), D_2(\mu_2), D_2(\mu_U)$ and d_2^{int} of two-way coupled Hénon maps (11) with different response systems ($C_1 = 0.5, B_1 = 0.3, B_2 = 0.1$). Additional line at 1.2 in the upper figure stands for the dimension of the attractor of Hénon system K. b) Normalized dimensions m_1^2, m_2^2 and m_U^2 for the same systems.

their asymptotic states are independent of their initial states, and depend only on the present state of the driver, it follows that $u_n = w_n$.

For $C_2 \geq 0.7$ the system M fully synchronizes with K, which leads to the collapse of the probability measure in K, M space on the diagonal (compare the discussion in the previous subsection).

(a)



(b)

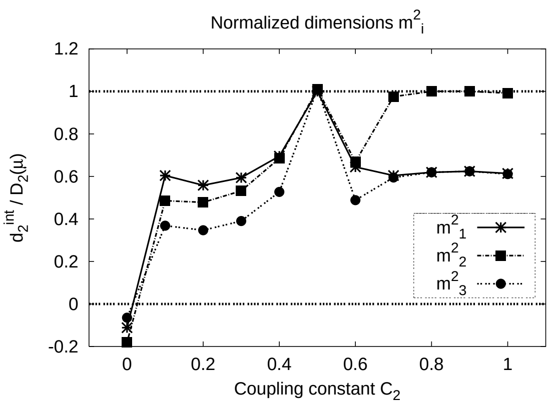


FIG. 7. a) Dimensions $D_2(\mu_1), D_2(\mu_2), D_2(\mu_U)$ and d_2^{int} of two-way coupled Hénon maps (11) with identical response systems ($C_1 = 0.5, B_1 = 0.3, B_2 = 0.3$). Additional line at 1.2 in the upper figure stands for the dimension of the attractor of Hénon system K. b) Normalized dimensions m_1^2, m_2^2 and m_U^2 for the same systems.

C. Logistic maps

Let $f_\alpha(x) := \alpha x(1-x)$. Consider a system consisting of four uncoupled logistic maps

$$x_{n+1}^i = f_{\alpha_i}(x_n^i),$$

where $\alpha_1 = 3.7$, $\alpha_2 = 3.8$, $\alpha_3 = 3.9$ and $\alpha_4 = 4$. Suppose the only variables available experimentally are[4] $Y^{i,j}(n) = F^{i,j}(x_n^i, x_n^j)$, $i < j$. Given two randomly chosen time series $Y^{i,j}(n), Y^{k,l}(n)$ we want to know if they share some degrees of freedom or not (if they "interact" or not). If i or j is equal to k or l, there are only three active degrees of freedom in the compound system. Otherwise there are four.

Estimated correlation dimensions for several cases are collected in Table 8. In every case we used time series 10^5 points long except for the last one, for which 10^6 points were used. The estimation error was roughly 2% except for the last case for which it was about 5-10%[5].

Consider now two symmetrically coupled logistic maps

$$\begin{cases} x_{n+1} &= f_\alpha(\tilde{x}_n), \\ y_{n+1} &= f_\beta(\tilde{y}_n), \end{cases} \quad \text{where} \quad \begin{cases} \tilde{x}_n &= \frac{x_n + c y_n}{1+c}, \\ \tilde{y}_n &= \frac{y_n + c x_n}{1+c}, \end{cases} \tag{12}$$

and parameter $c \in [0,1]$ measures the coupling. This is slightly different from couplings discussed previously in the literature (e.g. [54–56]). The maps are uncoupled for $c = 0$. For $c = 1$ (the strongest coupling) if we set $z_n := \tilde{x}_n = \tilde{y}_n$, we have $x_n = \frac{2\alpha}{\alpha+\beta} z_n$, $y_n = \frac{2\beta}{\alpha+\beta} z_n$, and $z_{n+1} = f_{\frac{\alpha+\beta}{2}}(z_n)$. Therefore dynamics is one-dimensional. Case $c > 1$ is equivalent to $c' = 1/c$.

Estimated correlation dimensions for several values of the coupling constant c are shown in figure 9. One can see the jump in the dimension of interaction from 0 at $c = 0$ to the value equal to the dimension of the whole systems

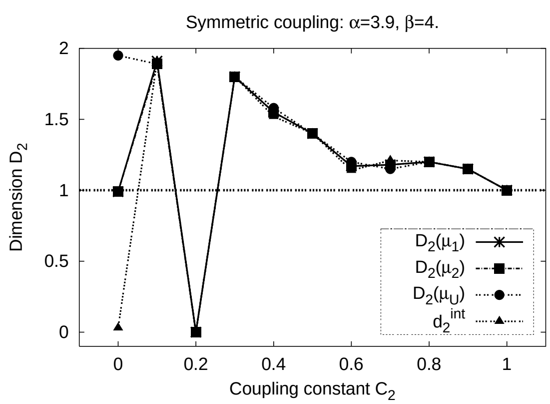
[4] The coupling functions $F^{i,j}$ were chosen randomly out of $x + y$, $x \cdot y$, $\sin(x)\cos(y)$, $x\exp(y)$, $2x - y$.

[5] We believe there are two reasons for this. One is higher dimensionality of the system in the last case (four uncupled logistic maps), the other is worse ergodicity in the phase space because the maps are uncoupled. Note that our procedure consists of two parts: first we make the embedding, then we calculate the dimensions. Each of the two can introduce errors. The number expected in the last case is the sum of the first four numbers, namely 3.87

series $x(n)$	$D(\mu_{x(n)})$
x_1	0.96
x_2	0.95
x_3	0.97
x_4	0.99
$Y^{1,2}$	1.88
$Y^{1,3}$	1.94
$Y^{1,4}$	1.95
$Y^{2,3}$	1.89
$Y^{2,4}$	1.94
$Y^{3,4}$	1.93
$f(Y^{1,2}, Y^{1,3})$	2.88
$f(Y^{1,2}, Y^{3,4})$	3.8

FIG. 8. Estimated correlation dimension for uncoupled logistic maps. The estimation error is roughly 2% except for the last number for which it is about 5-10%.

(a)



(b)

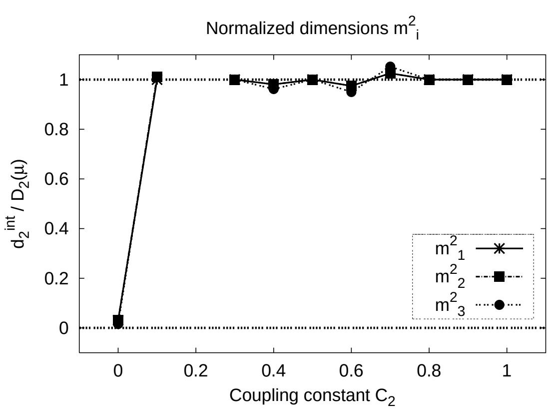


FIG. 9. a) Dimensions $D_2(\mu_1), D_2(\mu_2), D_2(\mu_U)$ and d_2^{int} of symmetrically coupled logistic maps (12). b) Normalized dimensions m_1^2, m_2^2 and m_U^2 for the same systems.

for positive c indicating case 4. in our classification. For $c = 0.2$ asymptotic dynamics settles on a periodic orbit leading to all the dimensions equal to 0. Numerically obtained approximations to asymptotic measures for coupling constant $c = 0., 0.1, 0.2, 0.3, 0.4, 0.5$ are shown in figure 10. Note the increasing synchronization between x and y.

It is of interest to compare the values of dimensions for $c = 0$ and 1, because in both cases $D_1(\mu_x) \approx D_1(\mu_y) = 1$, but the dimension of the whole system, estimated from $f(x_n, y_n)$ is equal to 2 in the first case, and 1 in the second, implying $D_{\mathsf{int}} = 0$ and 1 in these cases, respectively. Thus the first measure has a product structure, while the other is concentrated on the diagonal $x = y$.

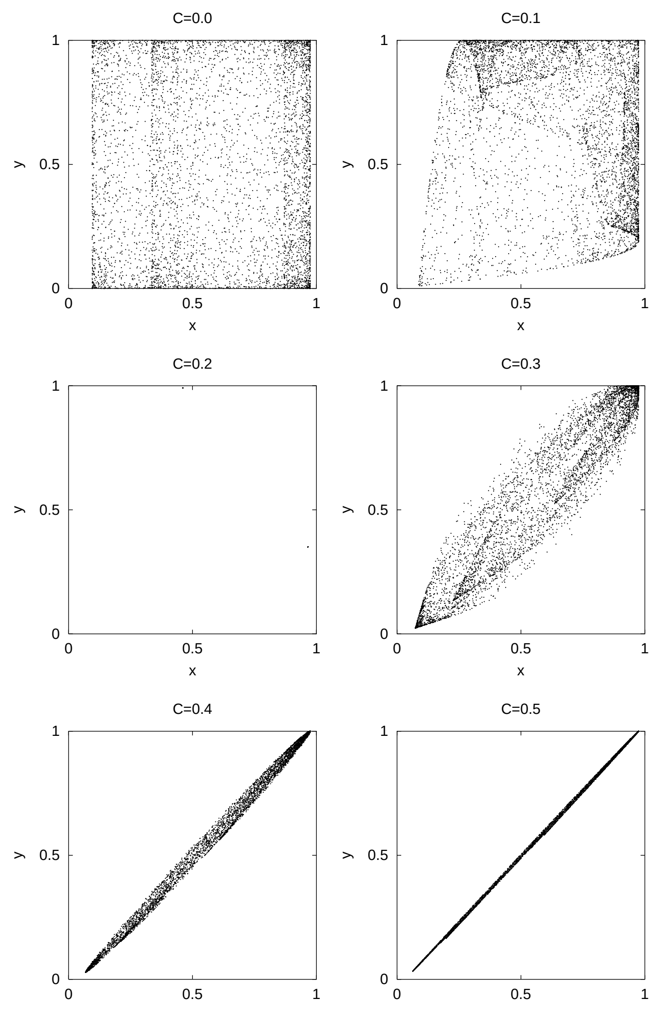


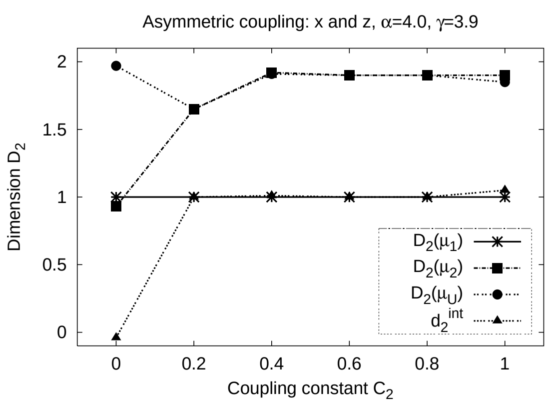
FIG. 10. Attractors of symmetrically coupled logistic maps (12) for $c = 0, 0.1, 0.2, 0.3, 0.4, 0.5$ in (x, y) plane.

The last case considered is that of the double control:

$$
\begin{cases} x_{n+1} = f_\alpha(x_n), \\ y_{n+1} = \frac{f_\beta(y_n)+c_1 x_n}{1+c_1}, \\ z_{n+1} = \frac{f_\gamma(z_n)+c_2 x_n}{1+c_2}, \end{cases} \tag{13}
$$

where $\alpha = 4.0$, $\beta = 3.8$, $\gamma = 3.9$, $c_1, c_2 \in [0,1]$. Let the observed systems U_1 and U_2 be the sets of all pairs (x,y) and (x,z), respectively. Then we have essentially the case 2. If U_1 and U_2 are the sets of all points x and pairs (x,z) then we have the case 3.

(a)



(b)

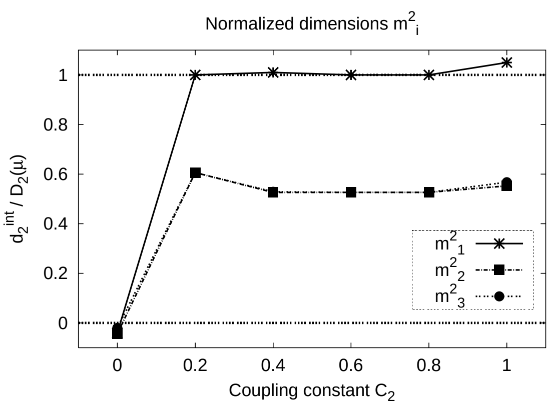
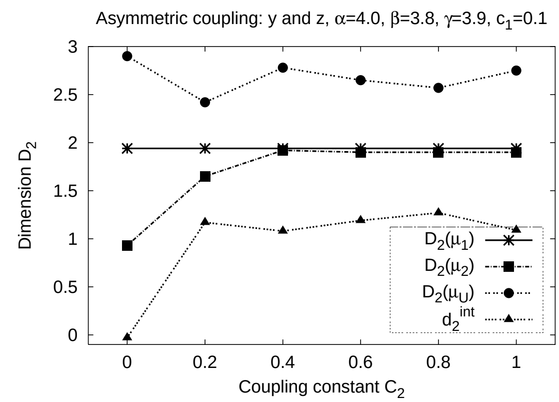


FIG. 11. a) Dimensions $D_2(\mu_1), D_2(\mu_2), D_2(\mu_U)$ and d_2^{int} of asymmetrically coupled logistic maps (13) when x and z are the observed variables. b) Normalized dimensions m_1^2, m_2^2 and m_U^2 for the same systems.

Figures 11 and 12 show estimated correlation dimension in these cases. Again, one can clearly see the difference between the coupled ($c_i > 0$) and uncoupled ($c_i = 0$) systems, because the interaction dimension jumps from 0 to 1 or more, in agreement with our expectations from theorems 2 and 3, since the dimension of the common part is 1 (x_n evolves according to Ulam map: $\alpha = 4.0$). Figure 13 shows projections of the attractor of (13) on (x,z) and (y,z) planes for $c_1 = 0.1$ and $c_2 = 0.2$.

(a)



(b)

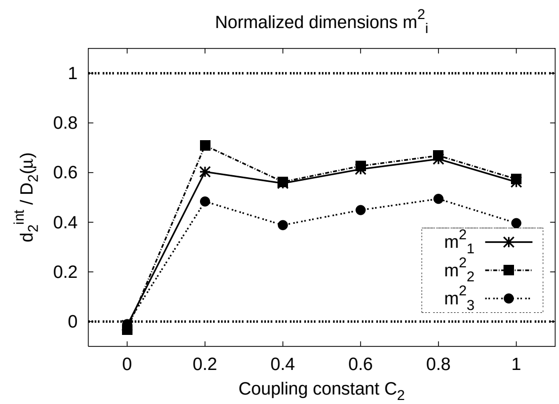


FIG. 12. a) Dimensions $D_2(\mu_1), D_2(\mu_2), D_2(\mu_U)$ and d_2^{int} of asymmetrically coupled logistic maps (13) when y and z are the observed variables. b) Normalized dimensions m_1^2, m_2^2 and m_U^2 for the same systems.

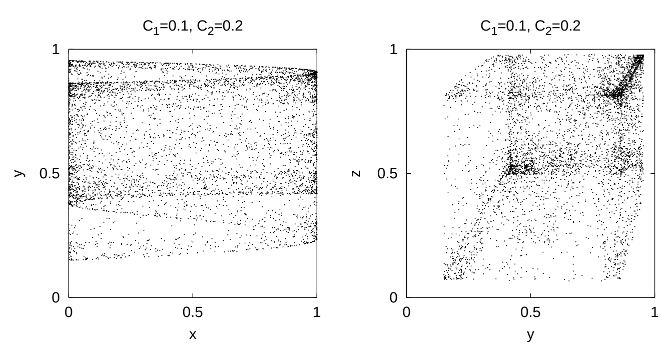


FIG. 13. Projections of the attractor of asymmetrically coupled logistic maps (13) for $c_1 = 0.1$, $c_2 = 0.2$ on (x,y) and (y,z) planes.

VI. CONCLUSIONS AND OUTLOOK

We presented a method which allows to distinguish interacting from non-interacting systems when time series of variables of the two systems are available. Partial proof of its validity was provided. Classification of all the possible interaction schemes was presented with examples of all the cases. Several simple interacting systems were analyzed.

To use our method in practice (from field data) we suggest the following procedure:

A. calculate the dimensions $D_q(\mu_1), D_q(\mu_2), D_q(\mu_U)$ and d_q^{int} (8) (we suggest $q = 1$ or $q = 2$; it is also good to normalize the data if they are od different orders);

B. repeat the calculation for several different coupling functions Y and average the results (linear combination seem to be the best choice);

C. if they are different from 0, calculate the normalized dimensions m_i^q (9).

D. they may take one, two or three distinct values.

   (a) if all of them are 0, the systems do not interact (case 1.);
   (b) if all of them are greater than 0 and less than 1, this is a generic case of interacting systems (case 4.);
   (c) if one of them is 1, the other are smaller, all the degrees of freedom of one system couple to some degrees of freedom of the other (case 3.), or we have the previous case (case 4.) but the variables of one of the systems which are not coupled to the other synchronize to the system comprising the common part of the dynamics;
   (d) if they are all equal to 1, all the degrees of freedom of one system couple to all the degrees of freedom of the other (case 2.), or we have the two previous cases (3. or 4.) but the variables of the two systems which are not coupled synchronize to the system comprising the common part of the dynamics.

Our method has been successfully used to distinguish between interacting and non-interacting Chua systems in an experiment [57]. We hope it shall prove a useful tool in analysis of other complex systems.

ACKNOWLEDGEMENTS

Discussions with several people enriched our understanding of the problem. In particular we want to thank Lou Pecora, Piotr Szymczak and Karol Życzkowski for illuminating comments. This work has been supported by the Polish Committee of Scientific Research under grant nr 2 P03B 036 16.

APPENDIX A: THE PROOFS.

Let μ_1, μ_2 be the invariant measures of systems U_1, U_2 as defined in Section II B.

Theorem 1 *Suppose $D_q(\mu_1), D_q(\mu_2), D_q(\mu_1 \times \mu_2)$ exist. Then*

$$D_q(\mu_1 \times \mu_2) = D_q(\mu_1) + D_q(\mu_2).$$

Proof: Take $q \neq 1$. For every $\varepsilon > 0$ consider partitions of $\mathbb{R}^{n_i}$ into cells of volume ε^{n_i}. This gives a partition in $\mathbb{R}^{n_1+n_2}$ into boxes of volume $\varepsilon^{n_1+n_2}$.

Let

$$\begin{aligned} p_j &= \mu_1(j-\text{th cell from the cover of } U_1), \\ r_k &= \mu_2(k-\text{th cell from the cover of } U_2). \end{aligned}$$

Then

$$\begin{aligned} D_q(\mu_1 \times \mu_2) &= \lim_{\varepsilon \to 0} \frac{1}{q-1} \frac{\log \sum_{k,j} p_i^q r_j^q}{\log \varepsilon} \\ &= \lim_{\varepsilon \to 0} \frac{1}{q-1} \frac{\log \left(\sum_k p_i^q\right) \left(\sum_j r_j^q\right)}{\log \varepsilon} \\ &= \lim_{\varepsilon \to 0} \left(\frac{1}{q-1} \frac{\log \sum_k p_k^q}{\log \varepsilon} \right) + \\ & \quad \lim_{\varepsilon \to 0} \left(\frac{1}{q-1} \frac{\log \sum_j r_j^q}{\log \varepsilon} \right). \end{aligned}$$

But the last two limits exist and are equal to $D_q(\mu_1)$ and $D_q(\mu_2)$, respectively.

The case of $q = 1$ is straightforward and left to the reader. □

For the next proof we need the following Lemma.

Lemma 5 Let $1 \geq c_{ij} \geq 0$, $\sum_{ij} c_{ij} = 1$, $a_i = \sum_j c_{ij}$, $b_j = \sum_i c_{ij}$. Then

$$\sum_{i,j} (a_i b_j \log(a_i b_j) - c_{ij} \log c_{ij}) \leq 0. \tag{A1}$$

Proof: Every convex function f satisfies Jensen's inequality

$$f\left(\sum_i p_i x_i\right) \leq \sum_i p_i f(x_i), \tag{A2}$$

where $\sum_i p_i = 1$. Since $f(x) = x \log x$ is convex, one has

$$\begin{aligned} f\left(\sum_{i,j} c_{ij}\right) &\leq \sum_{i,j} a_i b_j f\left(\frac{c_{ij}}{a_i b_j}\right), \\ f(1) &\leq \sum_{i,j} a_i b_j \frac{c_{ij}}{a_i b_j} (\log c_{ij} - \log a_i - b_j) \\ 0 &\leq \sum_{i,j} c_{ij} \log c_{ij} - \sum_{i,j} c_{ij} \log a_i - \sum_{i,j} c_{ij} \log b_j \end{aligned}$$

$$0 \leq \sum_{i,j} c_{ij} \log c_{ij} - \sum_i a_i \log a_i - \sum_j b_j \log b_j$$
$$0 \leq \sum_{i,j} (c_{ij} \log c_{ij} - a_i b_j \log(a_i b_j)),$$

where we took $p_{ij} = a_i b_j$ and $x_{ij} = c_{ij}/(a_i b_j)$. □

Let μ_1, μ_2, μ_x and μ_S be the invariant measures defined in Section II C.

Theorem 2 *Suppose $D_1(\mu_1)$, $D_1(\mu_2)$, $D_1(\mu_V)$, $D_1(\mu_U)$ exist. Then*

$$D_1(\mu_V) \leq d_{\mathsf{int}} := D_1(\mu_1) + D_1(\mu_2) - D_1(\mu_U).$$

(We shall call d_{int} dimension of interaction*). The equality holds when y_1 and y_2 are asymptotically independent.*

Proof: There are n_1+n_2 independent variables thus the system can be embedded in $\mathbb{R}^{n_1+n_2}$. Consider a partition of $\mathbb{R}^{n_1+n_2}$ into cells of size ε consistent with the structure of equations of dynamics, i.e. (i,j,k)-th cell$= A_i \times B_j \times C_k$, where A, B, C are ε-cells of dimension, respectively, $k_1 + k_2$, $n_1 - k_1$, $n_2 - k_2$ in spaces spanned by $\mathbf{x}$, $\mathbf{y}_1$ and $\mathbf{y}_2$.

Since the dynamics of $(\mathbf{x}, \mathbf{y}_1)$ is independent of $\mathbf{y}_2$, the invariant measure $\mu_1(A_i \times B_j)$ can be written as

$$\mu_1(A_i \times B_j) = \mu_V(A_i)\mu_{(y_1|x)}(B_j|A_i) =: p_i r_{ji},$$

where $\mu_{(y_1|x)}(B_j|A_i)$ are the conditional probabilities of finding the $\mathbf{y}_1$ in B_j under the condition $\mathbf{x}$ being in A_i. Similarly,

$$\mu_2(A_i \times C_k) = \mu_V(A_i)\mu_{(y_2|x)}(C_k|A_i) =: p_i s_{ki},$$

and

$$\begin{aligned}\mu_S(A_i \times B_j \times C_k) &= \mu_V(A_i)\mu_{(y_1,y_2|x)}(B_j, C_k|A_i). \\ &=: p_i t_{jki}\end{aligned}$$

If $\mu_{(y_1|x)}(B_j|A_i)$ and $\mu_{(y_2|x)}(C_k|A_i)$ are independent, then

$$\mu_{(y_1,y_2|x)}(B_j, C_k|A_i) = \mu_{(y_1|x)}(B_j|A_i)\mu_{(y_2|x)}(C_k|A_i), \tag{A3}$$

otherwise the only thing we know is that the l.h.s. measure is the coupling of the r.h.s. measures, namely

$$\sum_k \mu_{(y_1,y_2|x)}(B_j, C_k|A_i) = \mu_{(y_1|x)}(B_j|A_i),$$
$$\sum_j \mu_{(y_1,y_2|x)}(B_j, C_k|A_i) = \mu_{(y_2|x)}(C_k|A_i),$$

or

$$\sum_k t_{jki} = r_{ji},$$
$$\sum_j t_{jki} = s_{ki}.$$

Of course,

$$\sum_k s_{ki} = \sum_j r_{ji} = \sum_{jk} t_{jki} = 1,$$

if $p_i \neq 0$. Otherwise we take $\forall j,k:\ t_{jki} = 0$.

Taking this into consideration, inequality (5) follows:

$$\begin{aligned}
&D_1(\mu_1) + D_1(\mu_2) - D_1(\mu_V) - D_1(\mu_U) = \\
&\quad \lim_{\varepsilon\to 0} \frac{\sum_i \sum_j p_i r_{ji} \log(p_i r_{ji})}{\log\varepsilon} + \\
&\quad \lim_{\varepsilon\to 0} \frac{\sum_i \sum_k p_i s_{ki} \log(p_i s_{ki})}{\log\varepsilon} + \\
&\quad - \lim_{\varepsilon\to 0} \frac{\sum_i p_i \log(p_i)}{\log\varepsilon} + \\
&\quad - \lim_{\varepsilon\to 0} \frac{\sum_{i,j,k} p_i t_{jki} \log(p_i t_{jki})}{\log\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{\sum_i p_i \log(p_i) \left(\sum_j r_{ji} + \sum_k s_{ki} - 1 - \sum_{j,k} t_{jki}\right)}{\log\varepsilon} + \\
&\quad \lim_{\varepsilon\to 0} \frac{\sum_i p_i \sum_{j,k} (r_{ji} s_{ki} \log(r_{ji} s_{ki}) - t_{jki} \log(t_{jki}))}{\log\varepsilon} \\
&\geq 0
\end{aligned}$$

where in the last line we used Lemma 5 for $c = t$, $a = r$ and $b = s$ and the fact that $\log\varepsilon < 0$.

Note that the equality holds if and only if

$$t_{jki} = r_{ji} s_{ki}. \tag{A4}$$

This is what we call asymptotical independence of variables y_1 and y_2. In particular, when y_i are in generalized synchrony with x, this means that their asymptotic behavior is independent of their initial states and depends only on initial state of x, therefore their probability distributions cannot be independent, since they depend on the same number $x(0)$. However, we are not sure if this the only case when the equality is not satisfied, this is why we use another name for the above condition. □

One would like to establish a similar inequality in case of other Renyi dimensions, however, in general, even when (A4) is satisfied,

$$D_q(\mu_S) \neq D_q(\mu_1) + D_q(\mu_2) - D_q(\mu_x).$$

Indeed,

$$\begin{aligned}
&D_q(\mu_1) + D_q(\mu_2) - D_q(\mu_x) - D_q(\mu_S) = \\
&= \lim_{\varepsilon\to 0} \frac{1}{\log\varepsilon} \log\left[\frac{\left(\sum_{i,k} p_i^q r_{ki}^q\right)\left(\sum_{l,j} p_l^q s_{jl}^q\right)}{\left(\sum_l p_l^q\right)\left(\sum_{i,j,k} p_i^q s_{ji}^q r_{ki}^q\right)}\right] \\
&= \lim_{\varepsilon\to 0} \frac{1}{\log\varepsilon} \log\left[1 + \frac{\sum_{i<l,j,k} p_i^q p_l^q (r_{ki}^q - r_{kl}^q)(s_{ji}^q - s_{jl}^q)}{\sum_{i,j,k,l} p_i^q p_l^q s_{ji}^q r_{ki}^q}\right].
\end{aligned} \tag{A5}$$

This may have arbitrary sign and needs not vanish in the limit.

Although (A5) must go to 0 in the limit $q \to 1$, one can perhaps construct examples of measures for which the slope can be arbitrarily large. On the other hand, we believe such measures will not be typically observed in physical systems.

APPENDIX B: AN EXAMPLE OF PARTIALLY COUPLED SYSTEMS.

We present here a simple example of interacting systems for which one can introduce the natural decomposition (4).

Consider two systems U_1, U_2 interacting through a thin contact layer V. Denote variables in U_1 as $\mathbf{u}_1 = (\mathbf{v}_1, \mathbf{w}_1)$, variables in U_2 as $\mathbf{u}_2 = (\mathbf{v}_2, \mathbf{w}_2)$, and variables of the contact layer V are $(\mathbf{v}_1, \mathbf{v}_2)$. Dynamics of such a configuration can be described as

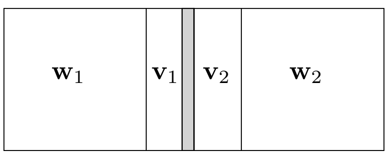


FIG. 14. Interacting systems.

$$\begin{aligned} \dot{\mathbf{w}}_1 &= f_1(\mathbf{v}_1, \mathbf{w}_1), \\ \dot{\mathbf{w}}_2 &= f_2(\mathbf{v}_2, \mathbf{w}_2), \\ \dot{\mathbf{v}}_1 &= g_1(\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_1), \\ \dot{\mathbf{v}}_2 &= g_2(\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_2). \end{aligned}$$

If we can average the influence of $\mathbf{w}_1, \mathbf{w}_2$ on the dynamics of $\mathbf{v}_1, \mathbf{v}_2$, e.g. when the time scales involved in the dynamics of $\mathbf{v}_i$ and $\mathbf{w}_i$ are different, we obtain

$$\begin{aligned} \dot{\mathbf{w}}_1 &= f_1(\mathbf{v}_1, \mathbf{w}_1), \\ \dot{\mathbf{w}}_2 &= f_2(\mathbf{v}_2, \mathbf{w}_2), \\ \dot{\mathbf{v}}_1 &= g_1(\mathbf{v}_1, \mathbf{v}_2, \lambda_1), \\ \dot{\mathbf{v}}_2 &= g_2(\mathbf{v}_1, \mathbf{v}_2, \lambda_2), \end{aligned} \tag{B1}$$

where λ_1, λ_2 measure the average influence of $\mathbf{w}_1, \mathbf{w}_2$ on V. Thus equations for $\mathbf{v}_1, \mathbf{v}_2$ comprise a closed system V. This part of dynamics is responsible for the interaction. Note that this scheme can also be considered as a double control configuration of three systems, where $(\mathbf{v}_1, \mathbf{v}_2)$ control $\mathbf{w}_1$ and $\mathbf{w}_2$.

If we set $\mathbf{x} := (\mathbf{v}_1, \mathbf{v}_2)$, $\mathbf{y}_i = \mathbf{w}_i$, then the equations (B1) reduce to equations (4).

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[1] N. H. Packard, J. P. Crutchfield, J. D. Farmer, and R. S. Shaw, Phys. Rev. Lett. **45**, 712 (1980).

[2] F. Takens, in *Dynamical Systems and Turbulence (Warwick 1980)*, Vol. 898 of *Lecture Notes in Mathematics ISBN 3 540 11171 9 and 0 387 11171 9*, edited by D. A. Rand and L.-S. Young (Springer-Verlag, Berlin, 1980), pp. 366–381.

[3] T. Sauer, J. A. Yorke, and M. Casdagli, J. Stat. Phys. **65**, 579 (1991).

[4] K. Suave, Consciousness and Cognition **8**, 213 (1999).

[5] G. Tononi and M. Edelman, Science **282**, 1846 (1998).

[6] R. F. Port and T. Van Gelder, *Mind as motion* (MIT Press, London, 1995).

[7] E. Thelen and L. B. Smith, *A dynamical systems approach to the development of cognition and action* (MIT presss, London, 1994).

[8] A. Nowak and R. R. Vallacher, *Dynamical social psychology* (Guilford, New York, 1998).

[9] *Dynamical systems in social psychology*, edited by R. R. Vallacher and A. Nowak (Academic Press, San Diego, 1994).

[10] H. Kantz and T. Schreiber, *Nonlinear Time Series Analysis* (Cambridge Univ. Press, Cambridge, UK, 1997).

[11] L. M. Pecora, T. L. Carroll, and J. F. Heagy, Phys. Rev. E **52**, 3420 (1995).

[12] S. J. Schiff *et al.*, Phys. Rev. E **54**, 6708 (1996).

[13] N. F. Rulkov, M. M. Sushchik, L. S. Tsimring, and H. D. I. Abarbanel, Phys. Rev. E **51**, 980 (1995).

[14] L. M. Pecora, T. L. Carroll, and J. F. Heagy, in *Nonlinear Dynamics and Time Series*, Vol. 11 of *Fields Inst. Communications*, edited by C. D. Cutler and D. T. Kaplan (American Math. Soc., Providence, Rhode Island, 1997), pp. 49–62.

[15] B. Mandelbrot, *The fractal geometry of nature* (Freeman, San Francisco, 1982).

[16] P. Meakin, *Fractals, scaling and growth far from equilibrium*, Vol. 5 of *Cambridge Nonlinear Science Series* (Cambridge University Press, Cambridge, 1998).

[17] Y. B. Pesin, *Dimension Theory in Dynamical systems: Contemporary Views and Aplications*, *Chicago Lectures in Mathematics* (Chicago University Press, Chicago and London, 1997).

[18] L. Olsen, Advances in Mathematics **116**, 82 (1995).

[19] A. Rényi, *Probability theory* (North-Holland, Amsterdam, 1971).

[20] P. Grassberger, Phys. Lett. A **97**, 227 (1983).

[21] H. G. E. Hentschel and I. Procaccia, Physica D **8**, 435 (1983).

[22] P. Grassberger and I. Procaccia, Physica D **9**, 189 (1983).

[23] P. Grassberger and I. Procaccia, Phys. Rev. Lett. **50**, 346 (1983).

[24] F. Hausdorff, Math. Annalen **79**, 157 (1919).

[25] J. D. Farmer, E. Ott, and J. A. Yorke, Physica D **7**, 153 (1983).

[26] J. D. Farmer, Z. Naturforsch. A **37**, 1304 (1982).

[27] R. Badii and A. Politi, Phys. Rev. Lett. **52**, 1661 (1984).

[28] R. Badii and A. Politi, J. Stat. Phys. **40**, 725 (1985).

[29] P. Grassberger, Phys. Lett. A **107**, 101 (1985).

[30] H. Kantz *et al.*, Phys. Rev. E **48**, 1529 (1993).

[31] T. C. Halsey, M. H. Jensen, L. P. K. I. Procaccia, and B. I. Shraiman, Phys. Rev. A **33**, 1141 (1986).

[32] K. J. Falconer, *Fractal geometry* (Wiley, Chichester, New

York, 1990).
[33] U. Frisch and G. Parisi, in *Turbulence and predictability in geophysical fluid dynamics and climate dynamics*, edited by M. Ghil, R. Benzi, and G. Parisi (North-Holland, New York, 1985), pp. 84–88.
[34] G. Paladin and A. Vulpiani, Phys. Rep. **156**, 147 (1987).
[35] T. Tél, Z. Naturforsch. A **43**, 1154 (1988).
[36] C. J. G. Evertsz and B. B. Mandelbrot, in *Multifractal Measures* (Springer, Berlin, New York, 1992), Chap. Appendix B, pp. 921–953.
[37] E. Ott, *Chaos in Dynamical Systems* (University Press, Cambridge, 1993).
[38] C. Beck and F. Schlögl, *Thermodynamics of Chaotic Systems* (Cambridge University Press, Cambridge, UK, 1997).
[39] K. Falconer, *Techniques in Fractal Geometry* (John Wiley and Sons, New York, 1997).
[40] J.-P. Eckmann and D. Ruelle, Rev. Mod. Phys. **57**, 617 (1985).
[41] L. Olsen, Math. Proc. Camb. Phil. Soc. **120**, 709 (1996).
[42] T. Sauer and J. A. Yorke, Int. J. of Bifurcation and Chaos **3**, 737 (1993).
[43] T. Sauer and J. Yorke, Ergodic Th. Dyn. Syst. **17**, 941 (1997).
[44] H. D. I. Abarbanel, R. Brown, J. L. Sidorowich, and L. S. Tsimring, Rev. Mod. Phys. **65**, 1331 (1993).
[45] H. D. I. Abarbanel, *Analysis of Observed Chaotic Data* (Springer-Verlag, New York Berlin Heidelberg, 1996).
[46] T. Schreiber, Phys. Rep. **308**, 2 (1999).
[47] M. Casdagli, S. Eubank, J. D. Farmer, and J. Gibson, Physica D **51**, 52 (1991).
[48] D. Wójcik, Ph.D. thesis, Center for Theoretical Physics, Polish Academy of Sciences, 2000, Warsaw, Poland; in preparation.
[49] R. Hegger, H. Kantz, and T. Schreiber, Chaos **9**, 413 (1999).
[50] F. Takens, in *Dynamical Systems and Bifurcations, Groningen 1984*, Vol. 1125 of *Lecture Notes in Mathematics*, edited by B. L. J. Braaksma, H. W. Broer, and F. Takens (Springer-Verlag, Berlin, 1985), pp. 99–106.
[51] J. Theiler, Phys. Lett. A **133**, 195 (1988).
[52] M. Hénon, Commun. Math. Phys. **50**, 69 (1976).
[53] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).
[54] A. S. Pikovsky and P. Grassberger, J. Physics A **24**, 4587 (1991).
[55] M. Zochowski and L. S. Liebovitch, Phys. Rev E **56**, 3701 (1997).
[56] *Handbook of Chaos Control*, edited by H. G. Schuster (Wiley-VCh, Weinheim, New York, 1998).
[57] D. Wójcik *et al.*, in preparation (unpublished).